Solving the Tower of Hanoi with Random Moves

Max A. Alekseyev* and Toby Berger†

April 16, 2013

Abstract

In this note we prove the exact formula for the expected number of moves to solve several variants of the Tower of Hanoi puzzle with 3 pegs and n disks, when each move is chosen uniformly from the set of all valid moves.

1 Introduction

We define two variants of the Tower of Hanoi puzzle with the following starting and final positions:

Puzzle 1. *The starting position is uniformly chosen from the set of all 3^n positions. The final position is with all disks on the same peg (no matter which one).*

Puzzle 2. *The starting position is with all disks on the first peg. The final position is with all disks on the third peg.*

Let $E_1(n)$ and $E_2(n)$ be the expected number of random moves required to solve Puzzle 1 and 2 respectively.

Puzzle 1 was posed by David G. Poole who submitted a sequence $E_1(n)$ for n up to 5 to the OEIS [7]. Later Henry Bottomley conjectured the following formula for $E_1(n)$:

$$E_1(n) = \frac{5^n - 2 \cdot 3^n + 1}{4}. \tag{1}$$

Puzzle 2 was posed by the second author [1] who also submitted to the OEIS [2] values of $E_2(n)$ for n up to 4 but did not conjecture a general formula for $E_2(n)$.

In this note we will prove the formula (1) as well as the following formula for $E_2(n)$:

$$E_2(n) = \frac{(3^n - 1)(5^n - 3^n)}{2 \cdot 3^{n-1}} \tag{2}$$

originally found by the first author.

Let D_k be the disk of size k such that in n-disk puzzle D_1 and D_n refer to the smallest and largest disks respectively. Similarly, let D_k^m, where $k \geq m$, be the set of all disks of sizes from m to k inclusively.

*Department of Computer Science and Engineering, University of South Carolina. Email: maxal@cse.sc.edu

†Department of Electrical and Computer Engineering, University of Virginia. Email: tb6n@virginia.edu

2 Auxiliary Puzzles

We start by defining two auxiliary puzzles:

Puzzle 3. *The starting position is with all disks on the first peg. The final position is with all disks on the same peg. At least one move is required.*

Let $E_3(n)$ be the expected number of moves required to solve this puzzle, while $p_1(n)$ and $p_2(n)$ denote the probability that a final configuration has all disks on the first and second peg, respectively.

From the symmetry, it is clear that the probability that a final configuration has all disks on the third peg is also $p_2(n)$, so $p_1(n) + 2p_2(n) = 1$.

Puzzle 4. *The starting position is with the largest disk on the second peg and the other disks on the first peg. The final position is with all disks on the same peg.*

Let $E_4(n)$ be the expected number of moves required to solve this puzzle, while $q_1(n)$, $q_2(n)$, and $q_3(n)$ denote the probability that a final configuration has all disks on the first, second, and third peg, respectively.

The relationship between Puzzles 3 and 4 and Puzzles 1 and 2 is given by the following theorems.

Theorem 1.

$$E_1(n) = E_1(n-1) + \frac{2}{3}E_4(n)$$

Proof. Note that D_n cannot move unless D_{n-1}^1 are on the same peg. In Puzzle 1, the expected number of steps required to arrive at such position is $E_1(n-1)$. Moreover, since the starting position is uniformly chosen, D_{n-1}^1 will be on any peg with the equal probability $1/3$. In particular, if it is the same peg where D_n resides, then the puzzle is solved. Otherwise, we can view the remaining moves as solving Puzzle 4, implying that

$$E_1(n) = \frac{1}{3}E_1(n-1) + \frac{2}{3}(E_1(n-1) + E_4(n)) = E_1(n-1) + \frac{2}{3}E_4(n).$$

□

Theorem 2.

$$E_2(n) = \frac{E_3(n)}{p_2(n)}$$

Proof. Note that D_n cannot move unless D_{n-1}^1 are on the same peg.

In solution of Puzzle 2 with random moves, first time all disks will appear on the same peg after $E_3(n)$ moves on average. This peg will be the first one with the probability $p_1(n)$, the second one with the probability $p_2(n)$, or the third one with the probability $p_2(n)$. In the last case the puzzle is solved, while in the first two cases we basically obtain a new Puzzle 2. Therefore, $E_2(n) = E_3(n) + (p_1(n) + p_2(n))E_2(n)$, implying that $E_2(n) = \frac{E_3(n)}{p_2(n)}$. □

Therefore, to find explicit formulae for $E_1(n)$ and $E_2(n)$, it is enough to find those for $E_3(n)$ and $E_4(n)$ along with the probability $p_2(n)$.

3 Derivation

Lemma 3. *The following equalities hold:*

(i) $E_3(n) = E_3(n-1) + 2p_2(n-1)E_4(n)$

(ii) $p_1(n) = p_1(n-1) + 2p_2(n-1)q_2(n)$

(iii) $p_2(n) = p_2(n-1)q_1(n) + p_2(n-1)q_3(n)$

(iv) $E_4(n) = \frac{1}{2} + E_3(n-1) + (p_1(n-1) + p_2(n-1))E_4(n)$

(v) $q_1(n) = p_1(n-1)q_1(n) + p_2(n-1)q_3(n)$

(vi) $q_2(n) = \frac{3}{4}(p_2(n-1) + (p_1(n-1) + p_2(n-1))q_2(n)) + \frac{1}{4}(p_1(n-1)q_3(n) + p_2(n-1)q_1(n))$

(vii) $q_3(n) = \frac{3}{4}(p_1(n-1)q_3(n) + p_2(n-1)q_1(n)) + \frac{1}{4}((p_1(n-1) + p_2(n-1))q_2(n) + p_2(n-1))$

Proof. Consider Puzzle 3. Note that D_n cannot move unless D^1_{n-1} are on the same peg. Therefore, we can focus only on D^1_{n-1} until they all come to the same peg, that will happen (on average) after $E_3(n-1)$ moves. When D^1_{n-1} are on the same peg, it is the first one (where D_n is) with the probability $p_1(n-1)$, in which case we have the final position with all disks on the first peg. If this is not the case (with the probability $1 - p_1(n-1) = 2p_2(n-1)$), we have D_n on the first peg and D^1_{n-1} on a different peg (equally likely on the second or the third one). Therefore, the remaining moves can be considered as Puzzle 4 and, hence, the expected number of remaining moves in this case is $E_4(n)$. Therefore, the expected total number of moves to solve Puzzle 3 is

$$p_1(n-1) \cdot E_3(n-1) + 2p_2(n-1) \cdot (E_3(n-1) + E_4(n)) = E_3(n-1) + 2p_2(n-1)E_4(n)$$

which proves formula (i).

From the above it is also easy to see that in the final position all disks will be at the first peg with the probability $p_1(n-1) + 2p_2(n-1)q_2(n)$ and at the second peg or third peg with the same probability $p_2(n-1)q_1(n) + p_2(n-1)q_3(n)$, that proves the formulae (ii) and (iii).

Now, consider Puzzle 4. Moves in this puzzle can be split into two or three stages as follows. In Stage 1 only D_n is moving (between the second and third pegs), Stage 2 starts with a move of D_1 (from the top of the first peg) and ends when D^1_{n-1} are on the same peg. If this is not the final position, the remaining moves are viewed as Stage 3. Let us analyze these stages.

It is clear that the expected number of moves in Stage 1 is $1/2$ while at the end of this stage we have D_n on the second peg with the probability $3/4$ and on the third peg with the probability $1/4$. The expected number of steps in Stage 2 is simply $E_3(n-1)$ and at the end D^1_{n-1} are on the first peg with the probability $p_1(n-1)$ and on the second or third pegs with the equal probability $p_2(n-1)$. Therefore, with the probability $p_2(n-1)$ we are at the final position (no matter where D_n is left after Stage 1) and with the probability $1 - p_2(n-1) = p_1(n-1) + p_2(n-1)$ we continue Stage 3. Stage 3 can be simply viewed as a new Puzzle 4 with the expected number of moves $E_4(n)$. The above analysis proves formulae (iv)-(vii). □

Note that the formula for $E_3(n)$ easily follows directly from (i) and (iv). Namely, the formula (iv) can be rewritten as $p_2(n-1)E_4(n) = \frac{1}{2} + E_3(n-1)$ and substituted into (i), resulting in the following recurrent formula

$$E_3(n) = E_3(n-1) + 2p_2(n-1)E_4(n) = 3E_3(n-1) + 1.$$

Together with $E_3(1) = 1$ it implies

$$E_3(n) = \frac{3^n - 1}{2} \quad \text{and} \quad E_4(n) = \frac{3^{n-1}}{2p_2(n-1)}.$$

Let us focus on the recurrent equations (ii), (iii), (v), (vi), (vii) and solve them with respect to $p_1(n)$, $p_2(n)$, $q_1(n)$, $q_2(n)$, and $q_3(n)$. Solving Puzzles 1 and 2 for $n = 2$, we have the following initial conditions:

$$p_1(2) = 5/8, \quad p_2(2) = 3/16, \quad q_1(2) = 1/8, \quad q_2(2) = 5/8, \quad q_3(2) = 1/4.$$

Also solving Puzzle 1 for $n = 1$, we get $p_1(1) = 0$ and $p_2(1) = 1/2$.

From (v) we have $p_2(n-1)q_3(n) = q_1(n) - p_1(n-1)q_1(n) = (1-p_1(n-1))q_1(n) = 2p_2(n-1)q_1(n)$ and since $p_2(n-1) > 0$ for all $n \geq 2$, we have $q_3(n) = 2q_1(n)$, $q_2(n) = 1-q_1(n)-q_3(n) = 1-3q_1(n)$, and $(8 - 3p_1(n-1))q_1(n) = 1$ for all $n \geq 2$.

Using these relations, we simplify equation (ii) to

$$\begin{aligned}
&p_1(n) = p_1(n-1) + 2p_2(n-1)(1 - 3q_1(n)) = 1 - 6p_2(n-1)q_1(n) \\
&= 1 - 3(1 - p_1(n-1))q_1(n) = 1 - 3q_1(n) + 3p_1(n-1)q_1(n) \\
&= 1 - 3q_1(n) + 8q_1(n) - 1 = 5q_1(n).
\end{aligned}$$

Combining the above equations, we have $(8 - 15q_1(n-1))q_1(n) = 1$, implying that $q_1(n) = f^{(n)}(0)$ (i.e., the n-th iteration of $f(\cdot)$) where

$$f(x) = \frac{1}{8 - 15x}.$$

It is easy to prove by induction that

$$q_1(n) = \frac{5^{n-1} - 3^{n-1}}{5^n - 3^n}.$$

Therefore,

$$q_2(n) = 1 - 3q_1(n) = \frac{2 \cdot 5^{n-1}}{5^n - 3^n};$$

$$q_3(n) = 2q_1(n) = 2\frac{5^{n-1} - 3^{n-1}}{5^n - 3^n};$$

$$p_1(n) = 5q_1(n) = \frac{5^n - 5 \cdot 3^{n-1}}{5^n - 3^n};$$

$$p_2(n) = \frac{1 - p_1(n)}{2} = \frac{3^{n-1}}{5^n - 3^n};$$

and finally

$$E_4(n) = \frac{3}{2}(5^{n-1} - 3^{n-1}).$$

Now we are ready to prove formulae (1) and (2). Theorem 1 together with $E_1(0) = 0$ implies

$$E_1(n) = \sum_{k=1}^{n} E_1(k) - E_1(k-1) = \sum_{k=1}^{n} 5^{k-1} - 3^{k-1} = \frac{5^n - 2 \cdot 3^n + 1}{4}.$$

Theorem 2 implies

$$E_2(n) = \frac{E_3(n)}{p_2(n)} = \frac{(3^n - 1)(5^n - 3^n)}{2 \cdot 3^{n-1}}.$$

4 One More Puzzle

Let us consider yet another puzzle somewhat similar to both Puzzles 1 and 2:

Puzzle 5. *The starting position is uniformly chosen from the set of all 3^n positions. The final position is with all disks on the first peg.*

Solving this Puzzle can be viewed as first solving Puzzle 1 and if it does not result in all disks on the first peg (that happens with the probability $2/3$), continue solving it as Puzzle 2. Therefore, the expected number of moves in Puzzle 5 is:

$$E_5(n) = E_1(n) + \frac{2}{3}E_2(n) = \frac{5^n - 2\cdot 3^n + 1}{4} + \frac{(3^n-1)(5^n-3^n)}{3^n} = \frac{5^{n+1} - 2\cdot 3^{n+1} + 5}{4} - \left(\frac{5}{3}\right)^n.$$

5 Analysis of Puzzle 2 via Networks of Electrical Resistors

We now present an alternative analysis of Puzzle 2 that couples a theorem about expected commute times of random walks on graphs with the delta-to-wye transformation originally introduced to aid in the analysis of three-phase AC systems for electrical power distribution. Building on a monograph by P. G. Doyle and J. L. Snell [4], A. K. Chandra et al. [3] proved the following theorem:

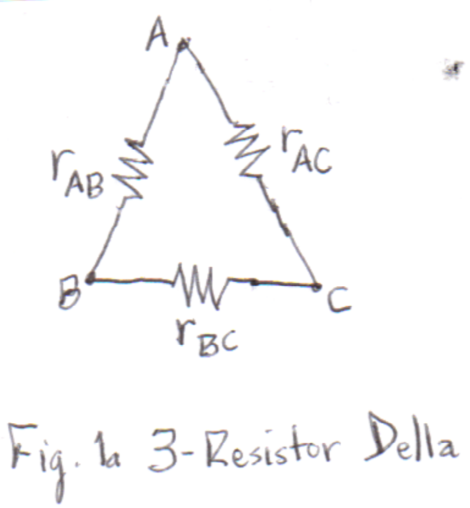
Theorem 4 (The Mean Commute Theorem)**.** *The expected number of steps in a cyclic random walk on an undirected graph that starts from any vertex A, visits vertex B, and then returns to A equals $2mR_{\mathrm{AB}}$, where m is the number of edges in the graph and R_{AB} is the electrical resistance between vertices A and B when a 1-ohm resistor is inserted in every edge of the graph.*

A Hanoi Tower with $n = 1$ may be represented by a graph with three nodes, which we shall denote by A, B, and C, and three egdes $\overline{\mathrm{AB}}, \overline{\mathrm{BC}}$, and $\overline{\mathrm{AC}}$. A, B and C correspond, respectively, to the three possible states, "D_1 is on peg 1", "D_1 is on peg 2", and "D_1 is on peg 3". Edge $\overline{\mathrm{AC}}$'s presence represents the fact that it is possible to reach C from A, or A from C, in one move; the other two edges have analogous interpretations. Now insert a 1 ohm resistor in each of these edges. There are two parallel paths between A and C, the direct path along edge $\overline{\mathrm{AC}}$ which has a resistance of 1 ohm, and the indirect path from A to B to C that has a resistance of 2 ohms, so the overall resistance from A to C is $2/3$ ohm. Since there are 3 edges in the graph, the mean commute time from A to C and back is $2 \cdot 3 \cdot 2/3 = 4$. By symmetry, on average half of the time is spent going from A to C and the other half returning from C to A. Accordingly, the mean time it takes a randomly moving Hanoi Tower with $n = 1$ to reach peg 3 starting from peg 1 is $4/2 = 2$, which agrees with our formula for $E_2(n)$ when $n = 1$.

We proceed to iterate this approach in order to derive the formula for $E_2(n)$ for general n. The key is to use the classical delta-to-wye transformation of electrical network theory [5]. A "delta" is a triangle with with vertices A, B and C that has resistances r_{AB} in edge $\overline{\mathrm{AB}}$, r_{AC} in edge $\overline{\mathrm{AC}}$, and r_{BC} in edge $\overline{\mathrm{BC}}$, as depicted in Fig. 1a. The corresponding "wye," shown in Fig. 1b, has the same three nodes - A,B, and C - plus a fourth node, x, and three edges $\overline{\mathrm{Ax}}, \overline{\mathrm{Bx}}$ and $\overline{\mathrm{Cx}}$ that contain respective resistances r_a, r_b and r_c. It is straightforward to verify that, if

$$r_a = \frac{r_{\mathrm{AB}}r_{\mathrm{AC}}}{r_{\mathrm{AB}}+r_{\mathrm{AC}}+r_{\mathrm{BC}}}, r_b = \frac{r_{\mathrm{AB}}r_{\mathrm{BC}}}{r_{\mathrm{AB}}+r_{\mathrm{AC}}+r_{\mathrm{BC}}} \text{ and } r_c = \frac{r_{\mathrm{AC}}r_{\mathrm{BC}}}{r_{\mathrm{AB}}+r_{\mathrm{AC}}+r_{\mathrm{BC}}}, \tag{3}$$

then the net resistance R_{AB} between nodes A and B will be the same in Fig. 1b as it is in Fig. 1a, and likewise for the net resistances R_{AC} between nodes A and C and R_{BC} between nodes B and C.



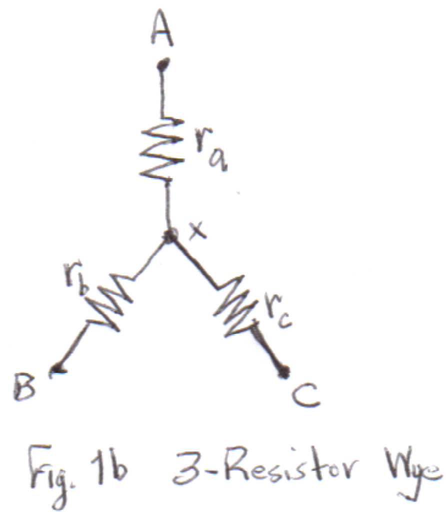


Figure 1:

We shall need to consider only the special case $r_{\mathrm{AB}} = r_{\mathrm{AC}} = r_{\mathrm{BC}} = R$, in which $r_a = r_b = r_c = R/3$. In particular, when $R = 1$, we have $r_1 = r_3 = 1/3$, so Fig. 1b yields $R_{\mathrm{AC}} = 2/3$, the same result we obtained before by considering the two parallel paths from A to C in Fig. 1a.

The graph for the state transitions of the Hanoi Tower for $n = 2$ is the Sierpinski gasket, or fractal, shown in Fig. 2. It is obtained by thrice replicating the graph for $n = 1$ and then connecting each of the resulting three pairs of non-extreme vertices by a bridging link. This leaves only three extreme vertices which we continue to label by A, B and C and which correspond to both disks being, respectively, on peg 1, on peg 2, and on peg 3. The other nodes, labeled $\alpha, \beta,\ \gamma, \delta, \lambda$ and μ, correspond respectively to the following ordered triples of contents of peg 1, peg 2 and peg 3: $\alpha = (D_2, \emptyset, D_1)$, $\beta = (D_2, D_1, \emptyset)$, $\gamma = (\emptyset, D_2, D_1)$, $\delta = (\emptyset, D_1, D_2)$, $\lambda = (D_1, D_2, \emptyset)$, $\mu = (D_1, \emptyset, D_2)$,

Theorem 5. *Delta-to-Wye Induction*
The Sierpinski gasket state diagram for an n-disk Hanoi Tower with a unit resistance in each of its branches can be converted, for purposes of determining the resistance between any two of the extreme vertices A, B and C, into a simple "wye" in which A, B and C each are connected to a centerpoint x, by links that each contain a common amount of resistance we shall denote by $R_n/2$.

Proof. We now prove Theorem 5 by induction. We have already shown that it is true for $n = 1$, the value of $R_1/2$ being $1/3$ ohm. We assume the theorem is true for n equal to the positive integer k and proceed to show that it them must also true for $n = k + 1$.

The Sierpinski gasket graph for a Hanoi Tower with $k+1$ disks is produced by generating three replicas of that for a Hanoi Tower with k disks that possess respective extreme vertices $\{A_1, B_1, C_1\}$. $\{A_2, B_2, C_2\}$ and $\{A_3, B_3, C_3\}$ and then adding three bridging links, as shown in Fig. 3, one between B_1 and A_2, another between C_1 and A_3, and the third between C_2 and B_3. The extreme vertices in the resulting graph are A_1, B_2 and C_3, i.e. the only A_i's, B_i's and C_i's to which none of the three

Fig 2. Sierpinski gasket state diagram for $n=2$.
$A=(D_2^1,\phi,\phi)$, $B=(\phi,D_2^1,\phi)$,
$C=(\phi,\phi,D_2^1)$, $\alpha=(D_2,\phi,D_1)$,
$\beta=(D_2,D_1,\phi)$, $\gamma=(\phi,D_2,D_1)$,
$\delta=(\phi,D_1,D_2)$, $\lambda=(D_1,D_2,\phi)$
$\mu=(D_1,\phi,D_2)$

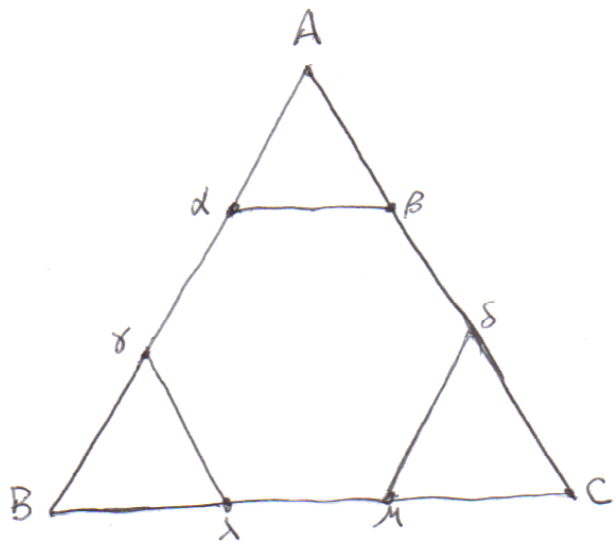


Figure 2:

bridging links is incident. When one seeks to apply Theorem 4, a unit resistance is inserted in each of the three bridging links, just as is the case for every other link in the graph. By the induction hypothesis, for purposes of computing the resulting resistances between any two of the vertices $A_1, B_1, C_1, A_2, B_2, C_2, A_3, B_3$, and C_3, the remainder of the resistive graph for $n = k+1$ consists of three wye's, each comprised of three links each of which have resistance ${}^{R_k}\!/_2$, as shown in Fig. 4. Denoting the centers of these three wye's by x, y, and z, we see that triange $\overline{\mathrm{xyz}}$ is a delta each edge of which has resistance ${}^{R_k}\!/_2 + 1 + {}^{R_k}\!/_2 = R_k + 1$. Applying the delta-to-wye transformation to this delta network results in the wye network of Fig. 5. Each of the links in this wye has the same resistance, namely

$$\frac{R_{k+1}}{2} = \frac{R_k}{2} + \frac{R_k + 1}{3} = \frac{5}{6}R_k + \frac{1}{3} \tag{4}$$

Theorem 5 is proved. □

In the process of proving Theorem 5, we have obtained the first order linear diffrence equation

$$R_{k+1} = \frac{5}{3}R_k + \frac{2}{3}. \tag{5}$$

From this equation and the boundary condition $R_1 = {}^2\!/_3$, we obtain the key result,

$$R_n = \left(\frac{5}{3}\right)^n - 1. \tag{6}$$

The number, m_n, of edges in the Sierpinski gasket graph for an n-disk Hanoi Tower is

$$m_n = \frac{(3^n - 3)\cdot 3 + 3\cdot 2}{2} = \frac{3}{2}(3^n - 1).$$

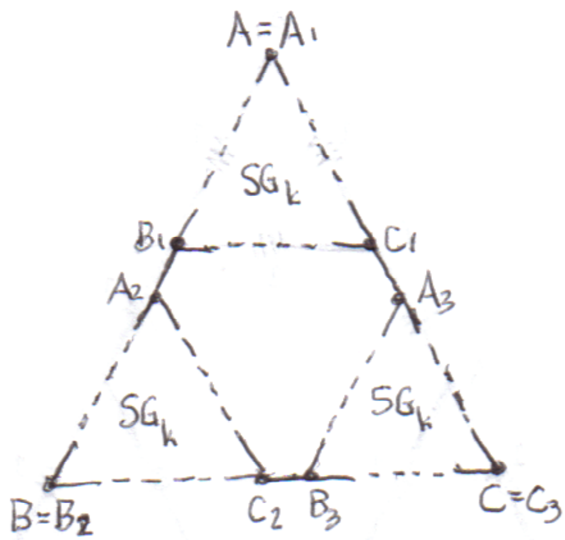


Figure 3. Sierpinski gasket SG_{k+1} for a Hanoi Tower with k+1 disks comprising three SG_k's and three bridging links $\overline{B_1A_2}$, $\overline{C_1A_3}$ and $\overline{C_2B_3}$.

Figure 3:

From the Mean Commute Time theorem and the symmetry of a random walks from A to C and from C to A, it follows that the mean number of steps it takes a randomly moving n-disk Hanoi Tower to transfer all its disks from peg 1 to peg 3 equals

$$m_n R_{\text{AC}} = m_n R_n = \frac{(3^n - 1)(5^n - 3^n)}{2 \cdot 3^{n-1}}. \tag{7}$$

This concludes our alternative derivation of formula (2) for the mean number $E_2(n)$ of moves in Puzzle 2.

6 Historical Commentary

The French mathematician, Edouard Lucas, created the Tower of Hanoi puzzle in 1883 [6]. Apparently, he simultaneously created the following legend [8]:

> "Buddhist monks somewhere in Asia are moving 64 heavy gold rings from peg 1 to peg 3. When they finish, the world will come to an end!"

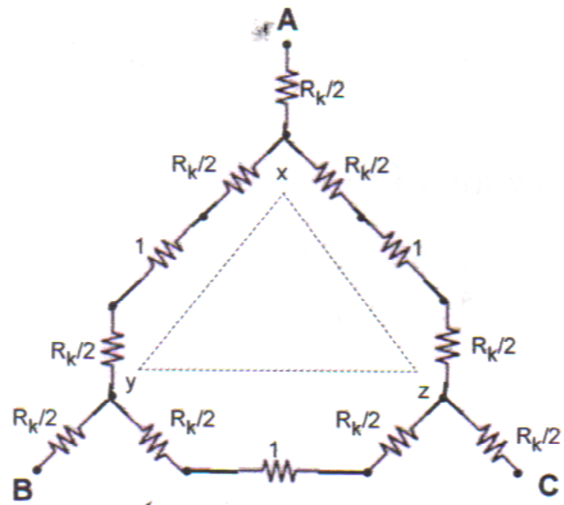
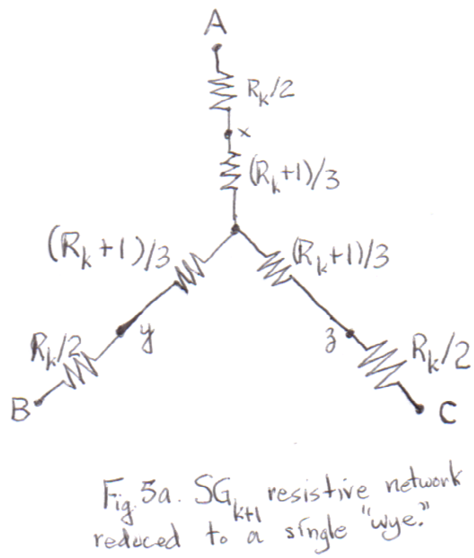


Fig. 4: A network of resistors apropos of a Hanoi Tower with *k+1* disks

Figure 4:

That Hanoi is located in what was then French Indo-China perhaps explains why a Frenchman saw fit to include Hanoi in the name of his puzzle. However, the legend never placed the monks and their tower explicitly in Hanoi or its immediate environs. Lucass Hanoi Tower puzzle became an international sensation (think Loyds 15 puzzle and Rubiks cube); the legend was used to bolster sales. Still a popular and beloved toy, the Hanoi Tower now also can be accessed over the Internet as a computer applet.

Since it is often intimated that monks possess superhuman abilities, maybe they can move large gold objects rapidly. Perhaps they can make a move a microsecond, maybe even a move a nanosecond. Considering that for $n = 64$ the minimum number of moves required to solve Puzzle 2 is "only" $2^{64}1 = 18,446,744,073,709,551,615$, planet Earth may expire any day now. This is in part what motivated suggesting adoption of a randomly moving Hanoi Tower [1]. Formula (7) shows that replacing the minimum-moves strategy with a random walk forestalls the end of the world by a factor of roughly $\left(\frac{5}{2}\right)^{64} > 2.9 \times 10^{25}$ on average. Although this is reassuring, it nonetheless would be further comforting to know that the coefficient of variation of the random number of steps in Puzzle 2 with $n = 64$ is small, i.e., that its standard deviation is many times smaller than its mean. Exact determination of said coefficient of variation is an open problem that we may address in future research. An extensive bibliography of some 370 mathematical articles concerning the Hanoi Tower puzzle and variations thereon has been complied by Paul Stockmeyer [9].



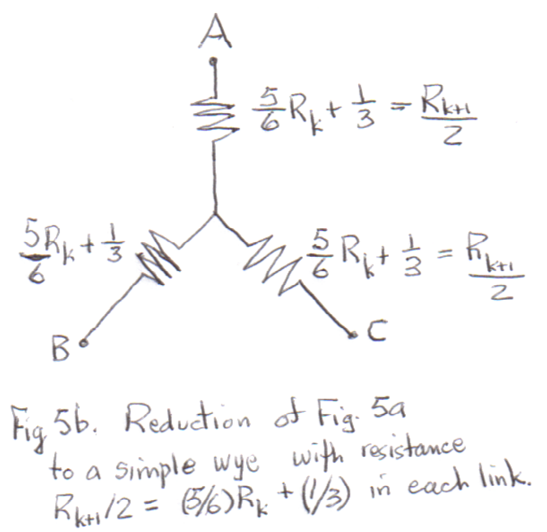


Figure 5:

References

[1] T. Berger. Lucas, Sierpinski, Markov, Shannon and The End of the World. In *Invited presentation in the David Slepian Memorial Session, Information Theory and Its Applications (ITA 2008)*, UCSD, La Jolla, CA, January 2008.

[2] Toby Berger. Sequence A134939 in N. J. A. Sloane (Ed.), The On-Line Encyclopedia of Integer Sequences. 2008. http://oeis.org/A134939.

[3] A. Chandra, P. Raghavan, W. L. Ruzzo, R. Smolensky, and P. Tiwari. The electrical resistance of a graph captures its commute and cover times. Seattle, May 1989. ACM Press.

[4] P. G. Doyle and J. L. Snell. *Random Walks and Electrical Networks.* The Mathematical Association of America, Washington, D.C., 1984.

[5] A. E. Kennelly. Equivalence of triangles and stars in conducting networks. *Electrical World and Engineer*, 34:413–414, 1899.

[6] E. Lucas. *Recreations Mathematiques*, volume III. Gauthier-Villiars, 1893. Reprinted by Albert Blanchard, Paris.

[7] David G. Poole. Sequence A007970 in N. J. A. Sloane (Ed.), The On-Line Encyclopedia of Integer Sequences. 2008. http://oeis.org/A007798.

[8] R. L. Ripley. *The New Believe It or Not Book 2nd Series.* Simon and Schuster, 1931.

[9] P. K. Stockmeyer. The Tower of Hanoi: A Bibliography. Version 2.2. 2005. http://www.cs.wm.edu/~pkstoc/biblio2.pdf.